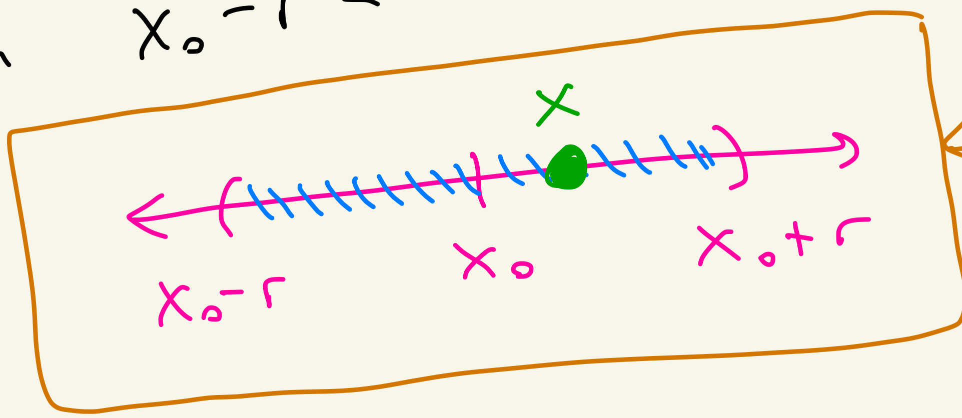
Topic 12 -
Power series solutions of ODEs

Def: We say that a function $f(x)$ is analytic at x_0 if we can write

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

and there is a radius of convergence $r>0$ where the series converges to $f(x)$ when $x_0 - r < x < x_0 + r$.



Ex: $f(x) = e^x$ is analytic at $x_0 = 0$

because

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \quad \text{when} \quad -\infty < x < \infty.$$

Ex: $f(x) = \ln(x)$ is analytic at $x_0 = 1$

because

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n \quad \text{when} \quad 0 < x < 2.$$

Ex: $f(x) = x^2$ is analytic at $x_0 = 2$

because

$$x^2 = \underbrace{4 + 4(x-2) + (x-2)^2}_{\text{finite power series}}$$

when $-\infty < x < \infty$

Ex: $f(x) = \dfrac{1}{1-x}$ is analytic at $x_0 = 0$

because

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{when} \quad -1 < x < 1.$$

In this topic, we will learn how to solve linear ODEs when the coefficients of the ODE are all analytic. We will use power series to do this

Theorem: Consider the initial-value problem

$$\boxed{\begin{gathered} y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = b(x) \\ y(x_0) = y_0,\ y'(x_0) = y_0',\ \dots,\ y^{(n-1)}(x_0) = y_0^{(n-1)} \end{gathered}}$$

If $a_{n-1}(x), \dots, a_1(x), a_0(x), b(x)$ are analytic at x_0, then there exists a unique solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \leftarrow \boxed{a_n = \frac{y^{(n)}(x_0)}{n!}}$$

that is analytic at x_0.

Moreover, if each of $a_{n-1}(x), \dots, a_1(x), a_0(x), b(x)$ have radii of convergence at least $r > 0$ at x_0, then $y(x)$ will at least have radius of convergence at least r at x_0

Ex: Consider the initial-value problem

$$y' - 2xy = 0, \quad y(0) = 1$$

$\boxed{x_0 = 0}$

Step 1: $y' + \underbrace{a_1(x)}_{-2x} y = \underbrace{b(x)}_{0}$

Here we have

$a_1(x) = -2x$

$b(x) = 0$

both of these are already power series centered at $x_0 = 0$. So have radius of convergence $r = \infty$ for both of them

Thus, $y' - 2xy = 0, \ y(0) = 1$ will have a unique solution

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n$$

that has radius of convergence $r = \infty$

Step 2: Find $y(x)$!

We use $y' - 2xy = 0, \ y(0) = 1$.

So we start with

$$\boxed{\begin{aligned} y' &= 2xy \\ y(0) &= 1 \end{aligned}}$$

Just keep differentiating the equation and keep plugging in $x_0 = 0$.

$$y' = 2xy$$
$$y'(0) = 2(0)\,y(0) = 2(0)(1) = 0$$

$y'(0) = 0$

$$y'' = 2y + 2xy'$$
$$y''(0) = 2y(0) + 2(0)\,y'(0)$$
$$= 2(1) + 2(0)(0)$$
$$= 2$$

$y''(0) = 2$

$$y''' = 2y' + 2y' + 2xy''$$
$$= 4y' + 2xy''$$
$$y'''(0) = 4y'(0) + 2(0)\,y''(0)$$
$$= 4(0) + 2(0)(2)$$
$$= 0$$

$y^{(3)}(0) = 0$

$$y^{(4)} = 4y'' + 2y'' + 2xy'''$$
$$= 6y'' + 2xy'''$$
$$y^{(4)}(0) = 6y''(0) + 2(0)\cdot y'''(0)$$
$$= 6(2) + 2(0)(0)$$
$$= 12$$

$y^{(4)}(0)$
$= 12$

$$y^{(5)} = 6y''' + 2y''' + 2xy^{(4)}$$
$$= 8y''' + 2xy^{(4)}$$
$$y^{(5)}(0) = 8y'''(0) + 2xy^{(4)}(0)$$
$$= 8(0) + 2(0)(12)$$
$$= 0$$

$y^{(5)}(0)$
$= 0$

$$y^{(6)} = 8y^{(4)} + 2y^{(4)} + 2xy^{(5)}$$
$$= 10y^{(4)} + 2xy^{(5)}$$
$$y^{(6)}(0) = 10y^{(4)}(0) + 2(0)\cdot y^{(5)}(0)$$
$$= 10(12) + 2(0)(0)$$
$$= 120$$

$y^{(6)}(0)$
$= 120$

So we have that

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

$$= 1 + 0 \cdot x + \frac{2}{2}x^2 + \frac{0}{3!}x^3 + \frac{12}{4!}x^4 + \frac{0}{5!}x^5 + \frac{120}{6!}x^6 + \dots$$

$$= 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \dots$$

Thus,

$$y(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \dots$$

solves

$$y' - 2xy = 0, \quad y(0) = 1.$$

Ex: Solve

$$y'' + x^2 y' - (x-1)y = \ln(x)$$

subject to $y'(1) = 0,\ y(1) = 0$

Step 1:

HW

$a_1(x) = x^2 = 1 + 2(x-1) + (x-1)^2 \leftarrow r = \infty$

$a_0(x) = -(x-1) \leftarrow r = \infty$

$b(x) = \ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \leftarrow r = 1$

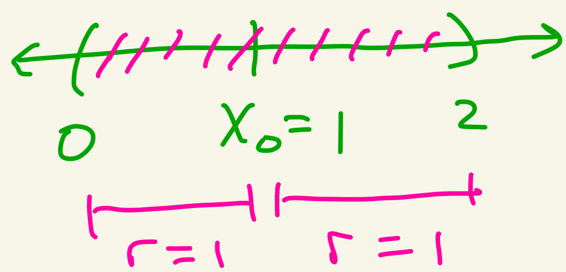
last topic

minimum of these is $r = 1$

Thus, the initial-value problem will have a power series solution

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{n!}(x-1)^n$$

that has radius of convergence at least $r = 1$, ie converges when $0 < x < 2$.



Step 2: Let's find $y(x)$!

We are given

$$\boxed{\begin{gathered} y'' + x^2 y' - (x-1)y = \ln(x) \\ y'(1) = 0,\ y(1) = 0 \end{gathered}} \leftarrow \boxed{\begin{gathered} y(1) = 0 \\ y'(1) = 0 \end{gathered}}$$

$$\boxed{\begin{aligned} y'' &= -x^2 y' + (x-1)y + \ln(x) \\ y''(1) &= -(1)^2 \cdot y'(1) + (1-1)\cdot y(1) + \ln(1) \\ &= -0 + 0 + 0 = 0 \end{aligned}} \leftarrow \boxed{y''(1) = 0}$$

$$\boxed{\begin{aligned} y''' &= -2xy' - 2x^2 y'' + y + (x-1)y' + \frac{1}{x} \\ y'''(1) &= -2(1)\underbrace{y'(1)}_{0} - 2(1)^2\underbrace{y''(1)}_{0} + \underbrace{y(1)}_{0} \\ &\quad + (1-1)\underbrace{y'(1)}_{0} + \frac{1}{1} \\ &= 1 \end{aligned}} \leftarrow \boxed{y'''(1) = 1}$$

$$y^{(4)} = -2y' - 2xy'' - 4xy'' - 2x^2y'''$$
$$+ y' + y' + (x-1)y' - x^{-2}$$

$$y^{(4)}(1) = -2\underbrace{y'(1)}_{0} - 2(1)\underbrace{y''(1)}_{0} - 4(1)\underbrace{y''(1)}_{0}$$
$$- 2(1)^2\underbrace{y'''(1)}_{1} + \underbrace{y'(1)}_{0} + \underbrace{y'(1)}_{0}$$
$$+ \underbrace{(1-1)y'(1)}_{0} - \underbrace{(1)^{-2}}_{1}$$

$$= -2 - 1 = -3$$

$$\boxed{y^{(4)}(1) = -3}$$

Thus, the first few terms of y's power series are

$$y(x) = y(1) + y'(1)(x-1) + \frac{y''(1)}{2!}(x-1)^2 + \frac{y'''(1)}{3!}(x-1)^3$$
$$+ \frac{y^{(4)}(1)}{4!}(x-1)^4 + \cdots$$
$$= 0 + 0(x-1) + 0(x-1)^2 + \frac{1}{6}(x-1)^3 - \frac{3}{24}(x-1)^4 + \cdots$$
$$= \frac{1}{6}(x-1)^3 - \frac{3}{24}(x-1)^4 + \cdots$$

and the series converges at least on $0 < x < 2$, ie with radius of convergence at least $r = 1$.